Effort complementarity and allocation of resources and roles in group contests

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Abstract

In this study, we consider multiple types of resource and role allocation design problems in a group contest with effort complementarity among group members. We propose an extended CES function form and its maximization for allocation to cover various forms of CES effort aggregator function, including those applied in previous studies. We show the allocation rules for multiple kinds of resources to group members, while previous studies handled only a single kind of resource. Our allocation rules imply that the more kinds of allocative resources the group manager holds, the stronger the effort complementarity the group members' cooperative work demands to maximize the winning probability. Furthermore, when the roles in the

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group are allocative, all resources and roles should be allocated to a single group member, even when each group member's effort is essential. We find that role allocation powerfully neutralizes effort complementarity.

Keywords: Group contest; CES effort aggregator function; Allocative resources and roles

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1 Introduction

In groups, we face the group members' free-rider problem. Once some group members contribute effort to their group's win in a contest among groups, the other group members can obtain the winning benefits without effort. How does a group manager who wants to maximize his/her group's winning probability in a group contest promote his/her group members' effort contributions to their win by mitigating the free-rider problem within his/her group? If he/she has nothing controllable in his/her group, he/she merely accepts their aggregated effort level in Nash equilibrium. If he/she has certain resources or roles controllable and allocative to his/her group members, the group manager uses them to promote his/her group members' effort contributions. In the real world, group managers usually allocate not only the prize, like a contingency fee, but also productive resources and roles. Consider R&D competitions among firms. In an R&D division in a firm, the division manager allocates a monetary prize to the group members conducting R&D when they win the competition. He/she also allocates equipment to facilitate experiments and roles in the experiments to his/her group members within the group to win the competition. The roles allocated to group members do not always result in the same effects on their effort contributions as the monetary prize because they are qualitatively different. The roles define how each group member's effort is converted into his/her group's impact on winning. A monetary prize is a winning benefit.

Here, if there is an effort complementarity among group members in their collaborative work, it reduces group members' free-riding incentives because their win depends on each individual's effort, which is not substitutive. The group manager has to take into account the properties of allocative resources and roles and the effort complementarity among the group members in addition to the differences in their abilities and effort costs in their cooperative work when he/she allocates them. The group manager may want to allocate all to a single group member with high ability, who can contribute much effort at low cost to the group, to benefit from his/her high performance without the free-rider problem. However, in the case in which the group members' efforts are strongly complementary to each other and essential, collaborative work may demand all group members' efforts. The group manager then needs to allocate the prizes and resources to all group members, depending on the degree of effort complementarity. Kolmar and Rommeswinkel (2013) show the relationship between effort complementarity and aggregated effort level by first using a CES production function as an aggregator of group members' efforts in the group contest literature. In this study, we show how to allocate multiple kinds of resources and roles among heterogeneous group members with different abilities in a stochastic group contest with the Tullock-form contest success function. We present the relationship between the group members' effort complementarity and the allocation rules using the CES effort aggregator, following Kolmar and Rommeswinkel (2013), which parameterizes the level of complementarity in a simple manner. We show that role allocation powerfully neutralizes effort complementarity, which is different from resource and prize allocations.

A few forms of the CES function and a form similar to it are used as the effort aggregators in the group contest literature (Epstein and Mealem 2009; Brookins, Lightle, and Ryvkin 2015; Choi, Chowdhury, and Kim 2016; Cheikbossian and Fayat 2018; Konishi and Pan 2020; Crutzen, Flamand, and Sahuguet 2020; Kobayashi and Konishi 2021.). Most of them use the simple CES function $(\sum_{k=1}^{n} e_k^{\sigma})^{\frac{1}{\sigma}}$, where e_k is a group member's effort, σ is a complementarity parameter, and n is the number of group members, as the effort aggregator. This CES function cannot cover $\sigma = 0$ even in the limit because of divergence. Brookins et al. (2015) use $(\frac{1}{n}\sum_{k=1}^{n} e_k^{\sigma})^{\frac{1}{\sigma}}$ (the form normalized by $1/n$) to cover $\sigma = 0$ in the limit. Kolmar and Rommeswinkel (2013) examined some variables using a general CES function form $g \cdot (\sum_{k=1}^{n} a_k(e_k)^{\sigma})^{\frac{1}{\sigma}}$.[1] Epstein and Mealem (2009) do not use a CES function, but a similar form, $\sum_{k=1}^{n} e_k^{\sigma}$. This form does not cover the strong complementarity of $\sigma \leq 0$. In this study, we use a CES function that covers a wider range, $-\infty < \sigma \leq 1$, than the simple CES function.

These previous studies handled a single kind of prize in the contest to promote group members' effort contributions. Kobayashi and Konishi (2021) clarify how to allocate the prize to group members to maximize the winning probability depending on the elasticity of effort cost and effort complementarity with reference to previous studies. However, as described above, in the real world, group managers allocate various kinds of resources and roles to their group members. The allocation rules depend on the effort-aggregator function form and the properties of the allocative resources and roles. To model these

[1] In Kolmar and Rommeswinkel (2013), when $\sum_{k=1}^{n} a_k = 1$, and in Brookins et al. (2015), the aggregators become a Cobb-Douglass function when $\sigma = 0$ in the limit.

elements theoretically, we need to tailor the maximization of the function to each effort aggregator function form and each property of allocative resources and roles. In this study, we propose an extended CES function form and its maximization to cover various CES effort aggregator functions, including the forms used in previous studies and various allocative resources and roles. Using the results of this maximization, we show the allocation rules of multiple types of resources and roles. Thanks to the wide coverage of effort complementarity, we can handle the case in which each group member's effort is strongly complementary and essential. By this method, we find that whether to allocate resources to all group members or to a single one decisively depends on the kinds of controllable variables; more precisely, whether the controllable variables include the roles or not. When the roles are not allocative and when the effort complementarity is weak, all resources should be allocated to only the group member with the highest ability in the group. When the roles are not allocative and when the effort complementarity is strong, the allocation rules of all kinds of resources follow the same rules, and the resources are allocated to all group members. The rules are the share of each group member's ability raised to the power of the relative complementarity to the elasticity of effort cost. From the form of rules, we find that the more controllable resources the group manager has, the stronger the effort complementarity the cooperative work by group members demands. When the group manager can control the roles of group members, the allocation of all kinds of resources and roles should be allocated to only the group member with the highest ability, even if each group member's effort has strong effort complementarity and is essential. This result indicates that the role allocation within the group represents the allocation of the essentialness of the respective group member's effort. This result also shows that no matter how strong the complementarity of each group member's effort, it cannot overcome the reduction in the aggregated effort level by group members' free riding when the roles are allocated.

The remainder of this paper is organized as follows. Section 2 presents the proposed model. In Section 3, we propose an extended CES function form and show the allocation rules of resources and roles to maximize the group's probability of winning. In Section 4, we consider our extended CES function form availability as an effort aggregator. Section 5 concludes the paper by discussing the applicability of our lemmas and future research. All proofs and calculation details are provided in the Appendix.

2 The model

We consider a contest in which $m \geq 2$ groups compete for a prize, focusing on a representative group $i = 1, 2, ..., m$. The population of group i is denoted by $n \geq 2$. Group members choose their effort levels e_j, $j = 1, 2, \ldots, n$, which contribute to their group's winning probability simultaneously and non-cooperatively. Group members' efforts are aggregated by the CES function $X_i = (\sum_{k=1}^{n} a_k (s_k e_k)^\sigma)^{\frac{1}{\sigma}}$, where $-\infty < \sigma \leq 1$ indicates the degree of effort complementarity.[2] $s_j > 0$ is group member j's skill, which converts

[2]Kolmar and Rommeswinkel (2013) call this effort aggregator an impact function.

his/her effort into the contribution to group i. a_j is the weight of j's contribution, which is viewed as the role assigned to him/her within the group, and we assume $\sum_{k=1}^{n} a_k = 1$. This CES aggregator becomes a linear function when $\sigma = 1$, becomes a Cobb-Douglas function when $\sigma = 0$ in the limit, and becomes a function with more effort complementarity among group members than the Cobb-Douglas function when $\sigma < 0$.[3] In the latter two cases, each member's effort is essential in the sense that if a member contributes no effort, the aggregate effort X_i is zero. In previous studies, σ is in a limited range, $\sigma > 0$ or $-\infty < \sigma \leq 1$, except for $\sigma = 0$ in a simplified CES form (Choi et al. 2016; Cheikbossian and Fayat 2018; Crutzen et al. 2020; Konishi and Pan 2020; Kobayashi and Konishi 2021). Our model covers a wider range of σ than previous studies to show the relation between σ and the control variables of the group manager.

The winning probability of group i is described as the Tullock-form contest success function $P_i = X_i/X$ where $X = \sum_{k=1}^{m} X_k$. We assume $P_i = 0$ when $X = 0$. The prize comprises divisible private goods that are shared among members of the winning group, and the value of the prize is normalized to 1. We denote the share of member j of group i by $v_j \in [0, 1]$. The share is allocated to each group member such that $\sum_{j=1}^{n_i} v_j = 1$. The effort cost function of group member j has a constant elasticity $\beta > 1$, that is, $e_j^{\beta}/(\beta c_j)$. $c_j = c_{1j} c_{2j} \cdots c_{dj}$, $d \geq 1$, is a composite of the limited resources that are allocated to group member j to reduce j's effort cost, such as IT equipment. If at least one type of $c_{dj} = 0$, j's

[3]The CES function goes to the Leontief function form as $\sigma \to -\infty$. In this study, we do not consider this case because it requires a different analysis. See Lee (2012).

marginal effort cost becomes infinite and he/she contributes nothing. Each kind of limited resource c_{dj} is essentially complementary for each group member. The expected payoff for member j of group i is $U_j = P_i v_j - e_j^\beta/(\beta c_j)$. We assume that each group member j regards all variables except for e_j as given and assume that all the above is common knowledge among all players. We employ the Nash equilibrium as the equilibrium concept in this group contest game.

Each group member decides his/her effort level to maximize his/her expected payoff, given the other group members and the other groups' aggregate effort $X - X_i$. The first-order condition of any member j of group i is, by using $P_i = X_i/X$,

$$P_i(1-P_i)\frac{e_j^{\sigma-1}}{X_i^\sigma}a_j s_j^\sigma v_j - \frac{e_j^{\beta-1}}{c_j} = 0.$$

We transpose $e_j^{\beta-1}/c_j$ to the other side, multiply this expression by $c_j/e_j^{\sigma-1}$, raise it to the power of $\sigma/(\beta-\sigma)$, multiply it by $a_j s_j^\sigma$, sum all $a_j(s_j e_j)^\sigma$ in group i, raise this expression to the power of $1/\sigma$, multiply it by $X_i^{\frac{\sigma}{\beta-\sigma}}$, and finally raise it to the power of $\beta - \sigma$. By using $X_i = P_i X$, the aggregate effort X_i at the Nash equilibrium within group i is implicitly described by

$$X^\beta = P_i^{1-\beta}(1-P_i)A, \tag{1}$$

where $A = \left[\sum_{k=1}^n (a_k^{\frac{\beta}{\sigma}} s_k^\beta c_k v_k)^{\frac{\sigma}{\beta-\sigma}}\right]^{\frac{\beta-\sigma}{\sigma}}$. A comprises exogenous variables for group members and is a CES function form, which is a Cobb-Douglass form when $\sigma = 0$ in the limit.[4] Note that c_j and v_j are homogeneous and interchangeable in A. Group i's winning

[4] $(a_j^{\frac{\beta}{\sigma}})^{\frac{\sigma}{\beta-\sigma}} = a_j^{\frac{\beta}{\beta-\sigma}} \to a_j$ as $\sigma \to 0$ in A, and recall $\sum_{k=1}^n a_k = 1$. Therefore, A converges to a

probability P_i is the share function of group i. We base the existence of a unique Nash equilibrium in this group contest game on the share function approach. See the details in Cornes and Hartley (2005). By totally differentiating (1) and substituting A of (1) into it, we obtain

$$\frac{dP_i}{dX} = \frac{\beta P_i(1-P_i)}{[(1-\beta)(1-P_i)-P_i]\,X} < 0, \tag{2}$$

that is, P_i is monotonically decreasing in X, and we have

$$\frac{dP_i}{dA} = \frac{-P_i(1-P_i)}{[(1-\beta)(1-P_i)-P_i]\,A} > 0. \tag{3}$$

In (1), $P_i \to 1$ as $X \to 0$ and $P_i \to 0$ as $X \to +\infty$ if $A > 0$, in addition to (2). There is then a unique X^*, where $\sum_{k=1}^{m} P_k = 1$, which means the existence of a unique Nash equilibrium in this group contest from the share function approach. Furthermore, for any $A < \hat{A}$, $P_i(X^*; A) < P_i(X^*; \hat{A})$ from (3). From this, $1 = \sum_{k\neq i} P_k(X^*) + P_i(X^*; A) < \sum_{k\neq i} P_k(X^*) + P_i(X^*; \hat{A})$, and $1 = \sum_{k\neq i} P_k(X^{**}) + P_i(X^{**}; \hat{A})$. Then, $X^* < X^{**}$ from (2). Therefore, $P_i(X^{**}; \hat{A}) = 1 - \sum_{k\neq i} P_k(X^{**}) > 1 - \sum_{k\neq i} P_k(X^*) = P_i(X^*; A)$. The last inequality means that an increase in A increases the share P_i (group i's winning probability) in the Nash equilibrium in this group contest. Therefore, by maximizing A, group i can maximize its winning probability in the Nash equilibrium. See Kobayashi and Konishi (2021) and Kobayashi, Konishi, and Ueda (2021) about the details of the above calculations.

Cobb-Douglass function.

3 Group manager's controllable variables

Suppose that there is a group manager who wants to maximize his/her group's winning probability P_i in each group. If the group manager has no controllable variable, he/she is only given the winning probability from the aggregate effort decided by (1) in the Nash equilibrium. If the group manager can allocate v_j, c_j, or a_j for $j = 1, \ldots, n$ to each group member, he/she can promote his/her group members' effort contributions by committing himself/herself to the allocation that maximizes A led in the last section before the contest. In this section, we show how to allocate these variables in relation to effort complementarity.[5] It should be noted that the form of A depends on the function forms of the effort aggregator and effort cost. When other variables are included in the CES effort aggregator or effort cost, A becomes another CES function form. To handle various CES forms of A, we consider the maximization of the extended CES function form

$$Y \equiv \left[\sum_{k=1}^{n} (\Pi_{h=1}^{q} y_{hk}^{\alpha_h})^{\rho} \right]^{\frac{1}{\rho}}$$

with t kinds of controllable variables $y_{1k}, y_{2k}, \ldots, y_{tk}$ for all $k = 1, \ldots, n$, such that $\sum_{k=1}^{n} y_{hk} = 1$ for all $h = 1, \ldots, t$, $1 \leq t \leq q$. We assume $-\infty < \rho \leq 1$ and $\rho = 0$ in the limit. In particular, we assume each $\alpha_g \geq 0$ for $g = 1, \ldots, t$ when $\rho \geq 0$. If $a_g < 0$

[5]We assume that each group manager cannot contract with each group member depending on either each member's effort level or their aggregated effort level. If such a contract is possible, the group manager can achieve his/her optimal effort level by offering the forcing contract to group members, as shown by Holmstrom (1982); that is, each group member has nothing if the aggregate effort level does not achieve the optimal level.

is also allowed when $\rho \geq 0$, we have $Y = +\infty$ for any $y_{hk} = 0$. We exclude this case. However, we allow $a_g < 0$ when $\rho < 0$. To consider the allocation rules, the following four lemmas and a corollary are useful:

Lemma 1. *Suppose $-\infty < \rho \leq 1$, $\rho \neq 0$, any $\alpha_g \geq 0$, and any $y_{hk} \geq 0$ in Y. The solution that maximizes Y with $y_{1k}, y_{2k}, \ldots, y_{tk}$ for all $k = 1, \ldots, n$ and $t = 1, \ldots, q$ such that $\sum_{k=1}^{n} y_{hk} = 1$ for all $h = 1, \ldots, t$ is as follows:*

1. *When $\rho \sum_{g=1}^{t} \alpha_g < 1$ or $\rho < \frac{1}{\sum_{g=1}^{t} \alpha_g}$, the solution is*

$$y_{1j}^* = y_{2j}^* = \ldots = y_{tj}^* = \frac{\Pi_{h=t+1}^{q} (y_{hj}^{\alpha_h})^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}}{\sum_{k=1}^{n} \Pi_{h=t+1}^{q} (y_{hk}^{\alpha_h})^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}} \tag{4}$$

for $j = 1, \ldots, n$. The maximized Y becomes the CES form

$$Y^* = \left[\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} \right]^{\frac{1-\rho \sum_{g=1}^{t} \alpha_g}{\rho}}. \tag{5}$$

When $t = q$,

$$y_{1j}^* = y_{2j}^* = \ldots = y_{tj}^* = \frac{1}{n}$$

and $Y^* = n^{\frac{1-\rho \sum_{g=1}^{q} \alpha_g}{\rho}}$.

2. *When $\rho \sum_{g=1}^{t} \alpha_g \geq 1$, a corner solution that is $y_{wj} = 1$ for all $w = 1, \ldots, t$ with the highest $\Pi_{h=t+1}^{q} y_{hj}^{\alpha_h}$, and the other $y_{wl} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ is obtained.*[6] *The maximized Y is $Y^* = \Pi_{h=t+1}^{q} y_{hj}^{\alpha_h}$. When $t = q$, $Y^* = 1$.*

[6]If there are two or more group members with the highest $\Pi_{h=t+1}^{q} y_{hj}^{\alpha_h}$ in Y, any one of them has $y_{wj} = 1$ and the others have $y_{wl} = 0$. The same applies in the following.

All proofs are provided in the appendix. In Lemma 1, we exclude both $\rho = 0$ and $\rho \to 0$. When $\rho = 0$, (4) in Lemma 1 becomes $1/n$. However, Y may be 0 or ∞. Lemma 1 is not applicable as it is to the case of $\rho = 0$ even in the limit. To handle $\rho \to 0$ in Lemma 1, we need an additional condition. Let $\alpha_r = (1+\rho)/\rho$ for only a single $r \in \{1, \ldots, q\}$ and $\sum_{k=1}^{n} y_{rk} = 1$ in Y.[7] As $\rho \to 0$,

$$Y \to \Pi_{k=1}^{n} (y_{rk} \Pi_{h \neq r} y_{hk}^{\alpha_h})^{y_{rk}}, \tag{6}$$

that is, a Cobb-Douglass function.[8] See the derivation of (6) in the appendix. In (6), if no y_{rk} is controllable, then Lemma 1 is applicable when $\rho = 0$ in the limit. Let $r = t+1$. We have the next lemma.

Lemma 2. *In Lemma 1, suppose* $\alpha_r = (1+\rho)/\rho$ *for* $r = t+1$ *and* $\sum_{k=1}^{n} y_{rk} = 1$. *As* $\rho \to 0$, *the solution* $y_{1j}^* = y_{2j}^* = \ldots = y_{tj}^* \to y_{rj}$ *for* $j = 1, \ldots, n$ *and*

$$Y^* \to \Pi_{k=1}^{n} \left(y_{rk}^{1+\sum_{g=1}^{t} \alpha_g} \Pi_{h=t+1, h \neq r}^{q} y_{hk}^{\alpha_h} \right)^{y_{rk}}. \tag{7}$$

From this lemma, Lemma 1 holds under the conditions of $\sum_{k=1}^{n} y_{rk} = 1$ and $a_r = (1+\rho)/\rho$ for $r = t+1$. Thus far, no y_{rk} for any k is controllable. We also consider the case in which each y_{rk} is added to the controllable variables as $r = t+1$ to maximize (6). The solution to this maximization then needs to be a corner solution of $y_{1j} = y_{2j} = \ldots = y_{tj} = y_{rj} = 1$ with the highest $\Pi_{h=t+2}^{q} y_{hj}^{\alpha_h}$ and the other $y_{hl} = 0$ for $h = 1, \ldots, t, t+1$

[7] When $\sum_{k=1}^{n} y_{rk} \neq 1$, Y is 0 or ∞ in the limit. See Appendix C of Kolmar and Rommeswinkel (2013).

[8] We obtain a Cobb-Douglass function $Y \to \Pi_{k=1}^{n} (\Pi_{h \neq r} y_{hk}^{\alpha_h})^{y_{rk}}$ as $\rho \to 0$ by simply setting $\alpha_r = 1/\rho$ for only a single $r \in \{1, \ldots, q\}$ and $\sum_{k=1}^{n} y_{rk} = 1$. $a_r = (1+\rho)/\rho$ is required in Proposition 1.

and $l = 1, \ldots, j-1, j+1, \ldots, n$. In fact, we differentiate (6) to the second order with regard to y_{rj} for every $j = 1, \ldots, n$ through the next calculation. Let $S = (y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}$. Then, $\log S = y_{rj}(\log y_{rj} + \log \Pi_{h\neq r}y_{hj}^{\alpha_h}) \Rightarrow S'/S = \log y_{rj} + \log \Pi_{h\neq r}y_{hj}^{\alpha_h} + 1 \iff S' = (\log y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h} + 1)(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}$.

$$\begin{aligned}\frac{\partial Y}{\partial y_{rj}} &= \frac{d(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}}{dy_{rj}}\Pi_{k\neq j}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}} \\ &= (\log y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h} + 1)(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_{hj}})^{y_{rj}}\Pi_{k\neq j}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}}.\end{aligned}$$

$$\frac{\partial^2 Y}{\partial y_{rj}^2} = \left[\frac{(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}}{y_{rj}} + (\log y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h} + 1)^2(\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}\right]\Pi_{k\neq j}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}} > 0. \tag{8}$$

For each y_{rj}, (6) is convex. Thus, to maximize the Cobb-Douglass function form of Y, $y_{rj} = 1$ with the highest $\Pi_{h=t+2}^{q}y_{hk}^{\alpha_h}$ and $y_{rl} = 0$, $l = 1, \ldots, j-1, j+1, \ldots, n$ is needed. Then, in Lemma 2, the other controllable variables y_{hl} raised to the power of $y_{rl} = 0$ need to be zero and $y_{hj} = 1$. However, such a corner solution cannot maximize (6) as it is because it includes the indeterminate form 0^0. Instead, when $y_{rj} = 1$ and $y_{sl} = 0$ in the limit in (7), we can obtain the corner solution.

Lemma 3. *In Lemma 2, suppose that y_{rk} is added to the controllable variables as $r = t+1$. The solution to the maximization of (7) is $y_{rj} = 1$ with the highest $\Pi_{h=t+2}^{q}y_{hj}^{\alpha_h}$, and the other $y_{rl} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ and $Y^* = \Pi_{h=t+2}^{q}y_{hk}^{\alpha_h}$ in the limit.*

Now we consider the final case in which y_{t+1k} with $\alpha_{t+1} < 0$ is added to the controllable variables when $\rho < 0$. Unfortunately, it may not be possible to maximize Y with respect to $y_{1k}, \ldots, y_{tk}, y_{t+1k}$ directly because Y again includes the indeterminate form when we have the corner solution. However, with regard to $y_{1k}, \ldots, y_{tk}$, case 1 of Lemma 1 always holds because $\rho \sum_{g=1}^{t} \alpha_g < 1$ always holds from $a_h > 0$ for $h = 1, \ldots, t$ when $\rho < 0$. Therefore, we can maximize Y^* in case 1 of Lemma 1 with y_{t+1k} involving $\alpha_{t+1} < 0$ for all $k = 1, \ldots, n$. Then, we have the following lemma:

Lemma 4. *Suppose $\rho < 0$ and suppose that y_{t+1k} with $\alpha_{t+1} < 0$ for all $k = 1, \ldots, n$ are the controllable variables in Y^* in Lemma 1. The solution maximizing Y^* with y_{t+1k} for all $k = 1, \ldots, n$ such that $\sum_{k=1}^{n} y_{t+1k} = 1$ is the following:*

1. *When $\rho \sum_{g=1}^{t+1} \alpha_g \leq 1$, a corner solution of $y_{t+1j}^{**} = 1$ with the highest $\Pi_{h=t+2}^{q} y_{hj}^{\alpha_h}$ and the other $y_{t+1l}^{**} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ is obtained. The maximized Y^* is $Y^{**} = \Pi_{h=t+2}^{q} y_{hj}^{\alpha_h}$.*

2. *When $\rho \sum_{g=1}^{t+1} \alpha_g > 1$, the solution is*

$$y_{t+1j}^{**} = \frac{\Pi_{h=t+2}^{q} (y_{hj}^{\alpha_h})^{\frac{\rho}{1-\rho \sum_{g=1}^{t+1} \alpha_g}}}{\sum_{k=1}^{n} \Pi_{h=t+2}^{q} (y_{hk}^{\alpha_h})^{\frac{\rho}{1-\rho \sum_{g=1}^{t+1} \alpha_g}}} \tag{9}$$

   for $j = 1, \ldots, n$. The maximized Y^* becomes the CES form

$$Y^{**} = \left[\sum_{k=1}^{n} \left(\Pi_{h=t+2}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t+1} \alpha_g}} \right]^{\frac{1-\rho \sum_{g=1}^{t+1} \alpha_g}{\rho}}. \tag{10}$$

In fact, when $\rho < 0$, $\alpha_{t+1} < 0$, $\rho \sum_{g=1}^{t} \alpha_g < 1$, $\rho \sum_{g=1}^{t+1} \alpha_g > 1$, and the conditions of Lemma 1 hold, then (4) becomes (9) by substituting (9) into (4). Besides, when $\rho < 0$, $\alpha_{t+1} < 0$, and $\rho \sum_{g=1}^{t+1} \alpha_g \leq 1$, by substituting the solution $y_{t+1j} = 1$ with the highest

$\Pi_{h=t+2}^{q} y_{hj}^{\alpha_h}$ and the other $y_{t+1l} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ in case 1 in Lemma 4 into (4), we have $y_{1j} = \ldots = y_{tj} = 1$ and $y_{1l} = \ldots = y_{tl} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ in Lemma 1, which correspond to those in Lemma 4. However, if we substitute this corner solution into Y directly, Y includes indeterminate forms. By maximizing (5) with y_{t+1k}, we can avoid these indeterminate forms despite the restricted maximization. This process is interpreted as a dynamic and backward maximization that maximizes Y with y_{t+1k}, predicting the second stage in the first stage, and that maximizes it with the other controllable variables $y_{1k}, \ldots, y_{tk}$ in the second stage. In addition, Lemma 4 holds even when $t = 0$. In Lemma 4, we maximize Y^* only with y_{t+1k}. This is possible without maximization with the other controllable variables y_{hk}, $h = 1, \ldots, t$ and given these variables. In fact, the proof of Lemma 4 holds when $t = 0$. Therefore, we have the following corollary:

Corollary 1. *Lemma 4 holds when also $t = 0$.*

Using these lemmas and corollary, we consider the group manager's maximization problem. Suppose that the group manager has t kinds of controllable variables of v_j and $c_j = c_{1j}c_{2j}\cdots c_{t-1j}$. Group members' cost reduction parameter c_j is a composite of multiple and complementary resources, including the case of a single kind of resource when $t = 2$. Each c_{hj} has a limited quantity in each group. The group manager maximizes A with t kinds of controllable variables $v_j, c_{1j}, \ldots, c_{t-1j}$ for $j = 1, \ldots, n$, such that $\sum_{k=1}^{n} v_k = 1$ and $\sum_{k=1}^{n} c_{hk} = 1$ for $h = 1, \ldots, t-1$. Recall the CES function form of A. These controllable variables are homogeneous and interchangeable in A. A one-unit change in v_j is the same as that of c_{hj} in A. Because each power of v_j and c_{hj} is 1 inside the CES function of A, this case is $q = d+3$, $\alpha_1 = \ldots = \alpha_{d+1} = 1$, $\alpha_{d+2} = \beta/\sigma$, $\alpha_{d+3} = \beta$, and $\rho = \frac{\sigma}{\beta-\sigma}$ in Lemma 1. $a_r = (1+\rho)/\rho = \frac{1+\sigma/(\beta-\sigma)}{\sigma/(\beta-\sigma)} = \beta/\sigma$ and $\sum_{k=1}^{n} y_{rk} = 1$ in Lemma 2 when $\sigma = 0$ in the limit. Therefore, in Y, $y_{1j}^{\alpha_1}, y_{2j}^{\alpha_2}, \ldots, y_{d+1j}^{\alpha_{d+1}}, y_{d+2j}^{\alpha_{d+2}}, y_{d+3,j}^{\alpha_{d+3}}$ are viewed as v_j, $c_{1j}, c_{2j}, \ldots, c_{dj}$, $a_j^{\frac{\beta}{\sigma}}$, and s_j^{β}, respectively. The power in (4) and (5) is $\frac{\rho}{1-\rho\sum_{g=1}^{t}\alpha_g} = \frac{\sigma/(\beta-\sigma)}{1-t\sigma/(\beta-\sigma)} = \frac{\sigma}{\beta-(1+t)\sigma}$. By substituting these variables into those in Lemmas 1 and 2, we obtain the following proposition:

Proposition 1. *Suppose that $-\infty < \sigma \leq 1$ in X_i and A. In a group contest among m groups, when the group manager can allocate the share of prize v_j and the cost reduction resources $c_{1j}, \ldots, c_{t-1j}$ for $j = 1, \ldots, n$ such that $\sum_{k=1}^{n} v_k = 1$ and $\sum_{k=1}^{n} c_{hk} = 1$ for $h = 1, \ldots, t-1$ to his/her group members, he/she should allocate them to maximize group i's winning probability as follows:*

1. *When $(t+1)\sigma < \beta$,*

$$v_j = c_{1j} = \ldots = c_{t-1j} = \frac{(a_j^{\frac{\beta}{\sigma}} s_j^{\beta}\ \Pi_{h=t}^{d} c_{hj})^{\frac{\sigma}{\beta-(t+1)\sigma}}}{\sum_{k=1}^{n}(a_k^{\frac{\beta}{\sigma}} s_k^{\beta}\ \Pi_{h=t}^{d} c_{hk})^{\frac{\sigma}{\beta-(t+1)\sigma}}}$$

for $j = 1, \ldots, n$ and

$$A^* = \left[\sum_{k=1}^{n}(a_k^{\frac{\beta}{\sigma}} s_k^{\beta}\ \Pi_{h=t}^{d} c_{hk})^{\frac{\sigma}{\beta-(t+1)\sigma}}\right]^{\frac{\beta-(t+1)\sigma}{\sigma}}.$$

As $\sigma \to 0$, $v_j = c_{1j} = \ldots = c_{t-1j} \to a_j$ for $j = 1, \ldots, n$ and $A^* \to \Pi_{k=1}^{n}\left(a_k^{1+t} s_k^{\beta}\ \Pi_{h=t}^{d} c_{hk}\right)^{a_k}$.

2. *When $(t+1)\sigma \geq \beta$, $v_j = c_{1j} = \ldots = c_{t-1j} = 1$ is allocated to the group member j with the highest $a_j^{\frac{\beta}{\sigma}} s_j^{\beta}\, \Pi_{h=t}^{d} c_{hj}$ and nothing is allocated to the other group members, that is, $v_l = c_{1l} = \ldots = c_{t-1l} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$. Then, $A^* = a_j^{\frac{\beta}{\sigma}} s_j^{\beta}\, \Pi_{h=t}^{d} c_{hj}$.*

In case 1 of Proposition 1, namely the strong complementarity case, by adding a single kind of c_{hj} to the controllable variables, c_{hj} itself disappears and t increases by one in each allocation rule and A^*. If every cost reduction resource c_j is controllable, then c_j itself disappears, and then $t = d+1$ and each allocation rule is the share of $(a_j^{\frac{\beta}{\sigma}} s_j^{\beta})^{\frac{\sigma}{\beta-(t+1)\sigma}} = (a_j s_j^{\sigma})^{\frac{\beta}{\beta-(t+1)\sigma}}$. That is basically the share of each group member's ability s_j with both the weight a_j and the power of the complementarity σ; however, the weighted ability becomes more powered as the complementarity becomes weaker, that is, as $(t+1)\sigma$ is close to β, and the share of the group member with the highest weighted ability goes to one. When the group manager has more controllable variables (higher t), this power effect is stronger. On the other hand, when the complementarity is stronger ($\sigma = 0$ in the limit), each share is equal to each weight a_j, and each ability becomes neutral. Furthermore, when the complementarity is much stronger ($\sigma < 0$), the shares of the group members with the high abilities become small because the group manager has to motivate group members with low abilities for their essentialized effort. $\sigma = 0$ is the threshold for whether each group member's ability enlarges or contracts each share. A multiplicity of controllable variables v_j and c_j amplifies the power effect in the allocation rule. Note that all allocation rules are $1/n$ in the case of $a_j = a_l$ and $s_j = s_l$ for any

$j, l = 1, \dots, n$.

With regard to the condition of $(t+1)\sigma < \beta$ (or $(t+1)\sigma \geq \beta$) in Proposition 1, it is a threshold condition whether the group manager allocates all resources to only a single group member or to all group members. Even if all weights and abilities are the same, this threshold holds. In addition, a multiplicity of controllable variables v_j and c_j also amplifies this threshold by $t\sigma$ under this condition. The more controllable variables the group manager holds (larger t), the lower the threshold of the complementarity becomes. In other words, when the group manager has many kinds of allocative resources, stronger effort complementarity (smaller σ) is needed for all group members' cooperative work. Under weak effort complementarity, even if sharing the prize among all group members is optimal for the group manager when only v_j is controllable, allocating all resources and a whole prize to a single group member can be optimal when v_j and c_j become controllable.

When $t = 1$, that is, the case of a single controllable variable, the condition $(t+1)\sigma < \beta$ (or $(t+1)\sigma \geq \beta$) becomes $2\sigma < \beta$ (or $2\sigma \geq \beta$) in the proposition. Previous studies have shown that the number of 2 is the threshold for effort complementarity or elasticity of effort cost. This threshold indicates whether adding group members heightens its winning probability and whether sharing the prize among all group members brings a higher winning probability than the monopolistic allocation to a single group member (Esteban and Ray 2001[9]; Epstein and Mealem 2009; Cheikbossian and Fayat 2018; Kobayashi and

[9] In Esteban and Ray (2001), the CES effort aggregator is not used, but rather a cost function with constant elasticity. In their results, 2 was also obtained as the threshold.

Konishi 2021). This threshold is critical in the free-rider problem among many group members, as Orson (1965) points out: the larger a group becomes, the more severe the free-rider problem. However, previous studies do not explain which property the number of 2 comes from. This threshold number comes from the maximization twice under the property of the constant elasticity of complementarity in the aggregator X_i and the effort cost: the first maximization of CES effort aggregator X_i with regard to group members' effort e_j and the second maximization of CES function form A with regard to the group manager's controllable variable v_j. The interior solutions in double maximization demand double σ less than the elasticity of effort cost. This means that allocating the prize to all group members is equivalent to adding new members to the group in terms of winning probability.

When c_{1j} is added to the group manager's controllable variable, the interior solutions in triple maximization demand triple σ less than the elasticity of effort cost, and so on. This threshold condition on σ in Proposition 1 is the concavity condition regarding the controllable variables v_j and c_j. From the standpoint of maximizing the winning probability, group members to whom these resources are allocated should not be expanded one by one as the effort complementarity is stronger, but should be expanded from a single group member to all group members at once at the threshold.

We consider the case in which a weight a_j is added to the controllable variables. Thus far, v_j and c_j are variables outside the CES aggregator X_i. a_j is inside it. This difference gives the different powers of the controllable variables in A. The group manager maximizes

A with $t+1$ kinds of controllable variables $v_j, c_{1j}, \ldots, c_{t-1j}$, a_j for $j = 1, \ldots, n$ such that $\sum_{k=1}^n v_k = 1$, $\sum_{k=1}^n c_{hk} = 1$ for $h = 1, \ldots, t-1$, and $\sum_{k=1}^n a_k = 1$. Let $\alpha_1 = \ldots = \alpha_t = 1$, $\alpha_{t+1} = \beta/\sigma$, $\alpha_{t+2} = \beta$, $q = d+3$, and $\rho = \frac{\sigma}{\beta-\sigma}$. $y_{1j}^{\alpha_1}, \ldots, y_{d+1j}^{\alpha_{d+1}}, y_{d+2j}^{\alpha_{d+2}}, y_{d+3j}^{\alpha_{d+3}}$ are viewed as v_j, $c_{1j}, c_{2j}, \ldots, c_{dj}$, $a_j^{\frac{\beta}{\sigma}}$, and s_j^β, respectively. $\rho \sum_{g=1}^{t+1} \alpha_g = \frac{\sigma}{\beta-\sigma}(t + \frac{\beta}{\sigma}) = \frac{\beta-\sigma+\sigma+t\sigma}{\beta-\sigma} = 1 + \frac{\sigma(t+1)}{\beta-\sigma}$. When $\sigma > 0$, $\rho \sum_{g=1}^{t+1} \alpha_g \geq 1$. Then, case 2 of Lemma 1 is applicable. When $\sigma = 0$ in the limit, Lemma 3 is applicable to the case as it is. When $\sigma < 0$, $\alpha_{t+1} = \beta/\sigma < 0$ and $\rho \sum_{g=1}^{t+1} \alpha_g < 1$. Then, case 1 of Lemma 4 and Corollary 1, in addition to the discussion just after Lemma 4, are applicable to this case. We have the next proposition.

Proposition 2. *In Proposition 1, suppose that a_j for $j = 1, \ldots, n$ is added to the controllable variables in A^*. The group manager should allocate it to maximize group i's winning probability as follows. When $t \geq 1$, $a_j = 1$ is allocated to the group member j with the highest $s_j^\beta\ \Pi_{h=t}^d c_{hj}$ and the other $a_l = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ and $A^{**} = s_j^\beta\ \Pi_{h=t}^d c_{hj}$ for any $-\infty < \sigma \leq 1$. Then all resources are allocated to group member j, $v_j = c_{1j} = \ldots = c_{t-1j} = 1$, and $v_l = c_{1l} = \ldots = c_{t-1l} = 0$ for the other members l. When $t = 0$, $a_j = 1$ is allocated to the group member j with the highest $s_j^\beta c_j v_j$, and the other $a_l = 0$ and $A^{**} = s_j^\beta c_j v_j$.*

When $\sigma < 0$, $\frac{\sigma}{\beta-\sigma}(t + \frac{\beta}{\sigma}) < 1$ in A. The form inside the brackets in A is concave. At a glance, the interior solution seems to maximize A. However, this is incorrect because the brackets are in the denominator of A; the corner solution maximizes A.

Suppose $t = d+1$ for a simple interpretation, as well as the discussion just after

Proposition 1. In the case in which the weight a_j is controllable, for any effort complementarity $-\infty < \sigma \leq 1$ and $\sigma = 0$ in the limit, the group manager should always allocate all resources to the group member with the highest ability s_j. This result indicates that even if each group member's effort is essential ($\sigma \leq 0$), the group manager should allocate all roles, resources, and prizes to this group member. What makes such an allocation possible is that the group manager can relieve the other group members with the lower abilities of their own roles and can assign those roles to this group member when the group manager controls the weight a_j. The role allocation neutralizes the essentialness of each group member's effort. When a_j is controllable, the number of controllable variables t does not affect the allocation of all to a single group member.

Even in the case in which the group manager can allocate only the roles, that is, he/she cannot concentrate a whole prize to a single group member, he/she should allocate all roles to a single group member for any effort complementarity. The other group members then completely free ride on this group member and receive their initial shares of the prize. Although such free riding happens, it is better for the group manager to concentrate all roles on the group member. These results show that effort complementarity cannot overcome the reduction in the aggregated effort level brought about by free riding, regardless of how strong it is when the roles are allocated.

We consider the peculiarity of the weight a_j from another standpoint. We add another controllable variable b_j such that $\sum_{k=1}^{n} b_k = 1$, which is inside the parentheses in the brackets of the CES effort aggregator function: $X_i = [\sum_{k=1}^{n} a_k (b_k s_k e_k)^{\sigma}]^{\frac{1}{\sigma}}$. b_j is viewed as a kind of allocative resource that works directly on the effort as well as the skill s_j. Then, $A = \left[\sum_{k=1}^{n}(a_k^{\frac{\beta}{\sigma}} b_k^{\beta} s_k^{\beta} c_k v_k)^{\frac{\sigma}{\beta-\sigma}}\right]^{\frac{\beta-\sigma}{\sigma}}$. The lemmas are also applicable to this case. Suppose $t+1$ kinds of controllable variables v_j, $c_{1j}, \dots, c_{t-1j}$, and b_k. The condition of concavity for the expression inside the brackets in A is $(t+\beta+1)\sigma < \beta$ instead of $(t+1)\sigma < \beta$ in Proposition 1. Then, the powers in the allocation rules and A^* in Proposition 1 are replaced by $\frac{\sigma}{\beta-(t+\beta+1)\sigma}$ and Proposition 1 holds. If a_j is added to the controllable variables, Proposition 2 holds. From these results, the behavior of b_j is similar to that of the controllable variables $c_{1j}, \dots, c_{t-1j}$, and v_j outside the CES of A. Comparing a_j with b_j as controllable variables inside the CES effort aggregator, the weight a_j has different properties from the other controllable variables inside and outside the CES effort aggregator. Once the weight is controllable, the work by a single group member is superior to the collaborative work by all group members in any effort complementarity, even when each group member's effort is essential. On the other hand, when only the other variables inside and outside the CES are controllable, the collaborative work by all group members is superior to the work by a single group member under strong effort complementarity. The control of weight a_j has a more powerful neutralization of the effort complementarity than any other variable.

4 Availability of Y as an effort aggregator

The form of Y can be used as an effort aggregator. By setting some variables in Y, we obtain the CES function forms used as the effort aggregator. For example, in Y, when $q = 3$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 1/\sigma$, $\rho = \sigma$, $y_{1j} = e_j$, $y_{2j} = s_j$, and $y_{3j} = a_j$, we have our effort aggregator $X_i = (\sum_{k=1}^{n} a_k(s_k e_k)^\sigma)^{\frac{1}{\sigma}}$. When $q = 1$, $\alpha_1 = 1$, $\rho = \sigma$, and $y_{1j} = e_j$ in Y, we have $X_i = (\sum_{k=1}^{n} e_k^\sigma)^{\frac{1}{\sigma}}$ in Choi et al. (2016), Cheikbossian and Fayat (2018), Crutzen et al. (2020), Konishi and Pan (2020), and Kobayashi and Konishi (2021). When $q = 3$, $\alpha_1 = 1$, $y_{1j} = g$, $\alpha_2 = 1/\sigma$, $y_{2j} = a_j$, $\alpha_3 = 1$, $y_{3j} = e_j$, and $\rho = \sigma$, for $j = 1, \ldots, n$, in Y, $X_i = g\left(\sum_{k=1}^{n} a_k(e_k)^\sigma\right)^{\frac{1}{\sigma}}$ in Kolmar and Rommeswinkel (2013) is obtained as well. In addition, Y becomes the effort aggregator function form $Y = \sum_{k=1}^{n} y_{1k}^\sigma$ of Epstein and Mealem (2009), when $q = 1$, $\rho = 1$, and $\alpha_1 = \sigma$. If there is a unique Nash equilibrium when Y is used as the effort aggregator, the form of Y unifies the various effort aggregators used in previous studies. Suppose $y_{1j} = e_j$ and $\alpha_1 > 0$ in Y as is used effort aggregator, that is, $X_i = [\sum_{k=1}^{n}(e_k^{\alpha_1}\Pi_{h=2}^{q} y_{hk}^{\alpha_h})^\sigma]^{\frac{1}{\sigma}}$. Recall $P_i = X_i/X$. We maximize group member j's utility $U_j = P_i v_j - e_j^\beta/\beta$ with e_j for $j = 1, \ldots, n$ by following the calculation process in Section 2. As a result, we have

$$X^{\frac{\beta}{\alpha_1}} = P_i^{\frac{\alpha_1-\beta}{\alpha_1}}(1 - P_i)\tilde{A}$$

where $\tilde{A} = \left[\sum_{k=1}^{n}\left(\alpha_1 v_k(\Pi_{k=2}^{q} y_{hk}^{\alpha_h})^{\frac{\beta}{\alpha_1}}\right)^{\frac{\alpha_1\sigma}{\beta-\alpha_1\sigma}}\right]^{\frac{\beta-\alpha_1\sigma}{\alpha_1\sigma}}$. The necessary and sufficient condition for $P_i \to 1$ as $X \to 0$ and $P_i \to 0$ as $X \to +\infty$ is $\alpha_1 < \beta$ if $\tilde{A} > 0$. Then, we also have

$$\frac{dP_i}{dX} = \frac{\beta P_i(1 - P_i)}{[(\alpha_1 - \beta)(1 - P_i) - \alpha_1 P_i]\, X} < 0,$$

and

$$\frac{dP_i}{d\tilde{A}} = \frac{-P_i(1-P_i)}{\left[\frac{\alpha_1-\beta}{\alpha_1}(1-P_i)-P_i\right]\tilde{A}} > 0.$$

There is then a unique Nash equilibrium from the share function approach, as in Section 2, and we can use Y as an effort aggregator under the condition of $\alpha_1 < \beta$. Therefore, we have the following proposition.

Proposition 3. *There is a unique Nash equilibrium in the group contest game with the effort aggregator* $X_i = [\sum_{k=1}^{n}(e_k^{\alpha_1}\Pi_{h=2}^{q} y_{hk}^{\alpha_h})^{\sigma}]^{\frac{1}{\sigma}}$ *when* $\alpha_1 > 0$ *and* $\alpha_1 - \beta < 0$.

From this result, Propositions 1 and 2 hold in the various CES functions and Epstein and Mealem's effort aggregators used in previous studies.

It should be noted that any CES form and Epstein and Mealem's form do not always yield the same results. In the CES effort aggregator, in the case of $2\sigma > \beta$ and 2σ close to β, while the group manager prefers a single member's work to group members collaborative work, the group member prefers collaborative work to his/her sole work. However, in Epstein and Mealem's form, this does not occur. Kobayashi and Konishi (2021) reported this result.

5 Concluding remarks

We conclude our paper by commenting on the applicability of our lemmas and the future research. The CES effort aggregators, including a similar function to the CES and the effort cost function with constant elasticity, have various forms, such as in previous studies and this study. The form of A in Section 2 depends on these forms. Owing to the lemmas on our extended CES function in this study, we can maximize the winning probability with various controllable variables, provided that A is a CES function form. Therefore, multiple kinds of variables can be pushed into CES aggregators and effort cost functions as product forms, depending on the various economic situations.

The maximization of the extended CES function form in this study may be applicable to other issues using a CES production function. As for the literature on group contests, a remaining issue in this study is to expand our model to dynamic allocation: The group manager decides each allocation sequentially. The condition may then change from the case of a simultaneous decision. While the allocation rules are the same in this study, they may be different in each stage in the case of sequential allocation. These are topics for future research.

Appendix

Proof of Lemma 1. When $\rho > 0$, maximizing the expression inside the brackets in Y, that is, maximizing Y^ρ, is equivalent to the maximization of Y. Y^ρ is a homogeneous function of degree $\rho \sum_{g=1}^{t} \alpha_g$ with regard to the controllable variables $y_{1k}, y_{2k}, \dots, y_{tk}$ and is additively separable. Then, $\rho \sum_{g=1}^{t} \alpha_g < 1$ is necessary for an interior solution to this maximization. When $\rho < 0$, minimizing Y^ρ is equivalent to maximizing Y because the brackets in Y become a denominator. Then, an interior solution for this minimization is obtained. In fact, any corner solution that includes $y_{wl} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$ yields $Y^\rho = +\infty$ and $Y = 0$, because each term $\Pi_{h=1}^q y_{hk}^{\alpha_h}$ is the denominator. Therefore, when $\rho \sum_{g=1}^t \alpha_g < 1$, the following procedure is applicable to both $\rho > 0$ and $\rho < 0$. The Lagrange function is defined as

$$L = \left[\sum_{k=1}^n (\Pi_{h=1}^q y_{hk}^{\alpha_h})^\rho \right]^{\frac{1}{\rho}} + \sum_{h=1}^t \lambda_h \left[1 - \sum_{k=1}^n y_{hk} \right].$$

The first-order conditions are

$$\frac{\partial L}{\partial y_{sj}} = \left[\sum_{k=1}^n (\Pi_{h=1}^q y_{hk}^{\alpha_h})^\rho \right]^{\frac{1}{\rho}-1} (\Pi_{h \neq s} y_{hj}^{\alpha_h})^\rho \alpha_s y_{sj}^{\alpha_s \rho - 1} - \lambda_s = 0$$

$$\frac{\partial L}{\partial \lambda_s} = 1 - \sum_{k=1}^n y_{sk} = 0$$

for $s = 1, \ldots, t$ and $j = 1, \ldots, n$. From any $\partial L / \partial y_{sj}$ and $\partial L / \partial y_{sl}$, we have

$$\frac{y_{sj}}{y_{sl}} = \left(\frac{\Pi_{h \neq s} y_{hj}^{\alpha_h}}{\Pi_{h \neq s} y_{hl}^{\alpha_h}} \right)^{\frac{\rho}{1 - \alpha_s \rho}}.$$

By choosing any u among $s = 1, \ldots, t$ and substituting y_{uj}/y_{ul} into y_{sj}/y_{sl}, we have

$$\frac{y_{sj}}{y_{sl}} = \left(\frac{\Pi_{h \neq s,u} y_{hj}^{\alpha_h}}{\Pi_{h \neq s,u} y_{hl}^{\alpha_h}} \right)^{\frac{\rho}{1 - \alpha_s \rho - \alpha_u \rho}}.$$

Repeating these choice and substitution for all $1, \ldots, t$, we have

$$\frac{y_{sj}}{y_{sl}} = \left(\frac{\Pi_{h=t+1}^q y_{hj}^{\alpha_h}}{\Pi_{h=t+1}^q y_{hl}^{\alpha_h}} \right)^{\frac{\rho}{1 - \rho \sum_{g=1}^t \alpha_g}}.$$

This expression is

$$y_{sl} = \left(\frac{\Pi_{h=t+1}^q y_{hl}^{\alpha_h}}{\Pi_{h=t+1}^q y_{hj}^{\alpha_h}} \right)^{\frac{\rho}{1 - \rho \sum_{g=1}^t \alpha_g}} y_{sj}.$$

By substituting y_{sl} for all $l = 1, \ldots, n$ into $\partial L / \partial \lambda_s$ and solving it for y_{sj}, we obtain (4):

$$
\begin{aligned}
\sum_{k \neq j} \left(\frac{\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h}}{\Pi_{h=t+1}^{q} y_{hj}^{\alpha_h}} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} y_{sj} + y_{sj} &= 1 \\
\frac{y_{sj}}{\left(\Pi_{h=t+1}^{q} y_{hj}^{\alpha} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}} \sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} &= 1 \\
y_{sj} &= \frac{\left(\Pi_{h=t+1}^{q} y_{hj}^{\alpha} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}}{\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}}
\end{aligned}
$$

for $s = 1, \ldots, t$ and $j = 1, \ldots, n$. When $t = q$, $y_{sj}/y_{sl} = 1$ in the above expression because all y_{hk} in Y are controllable variables and $\Pi_{h=t+1}^{q} y_{hk}$ becomes an empty product and becomes 1. Thus, $y_{sj} = 1/n$ for any j and s.

Furthermore, substituting all (4) into Y, we obtain the maximized Y, that is, (5):

$$
\begin{aligned}
Y^* &= \left[\sum_{k=1}^{n} \left(\Pi_{f=1}^{t} \left(\frac{\left(\Pi_{h=t+1}^{q} y_{hj}^{\alpha} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}}{\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}}} \right)^{\alpha_f} \Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\rho} \right]^{\frac{1}{\rho}} \\
&= \frac{1}{\left(\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} \right)^{\sum_{g=1}^{t} \alpha_g}} \left[\sum_{k=1}^{n} \left(\left(\Pi_{h=t+1}^{q} y_{hj}^{\alpha} \right)^{\frac{\rho \sum_{h=1}^{t} \alpha_h}{1-\rho \sum_{g=1}^{t} \alpha_g}} \Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\rho} \right]^{\frac{1}{\rho}} \\
&= \frac{1}{\left(\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} \right)^{\sum_{g=1}^{t} \alpha_g}} \left[\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hj}^{\alpha} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} \right]^{\frac{1}{\rho}} \\
&= \left[\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} \right)^{\frac{\rho}{1-\rho \sum_{g=1}^{t} \alpha_g}} \right]^{\frac{1-\rho \sum_{g=1}^{t} \alpha_g}{\rho}} .
\end{aligned}
$$

When $t = q$, $Y^* = n^{\frac{1-\rho \sum_{g=1}^{q} \alpha_g}{\rho}}$, because $\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h} = 1$ inside the parentheses in the brackets in Y^*.

When $\rho \sum_{g=1}^{t} \alpha_g \geq 1$, Y^ρ is a homogeneous function of degree 1 or more. Noting that Y^ρ is additively separable, we have a corner solution of $y_{wj} = 1$ for all $w = 1, \ldots, t$ with the highest $\Pi_{h=t+1}^{q} y_{hj}^{\alpha_h}$ and the other $y_{wl} = 0$ for $l = 1, \ldots, j-1, j+1, \ldots, n$. Then, $Y^* = \Pi_{h=t+1}^{q} y_{hj}^{\alpha_h}$. When $t = q$, $Y^* = 1$. ■

Deriving (6). Let $a_r = (1+\sigma)/\sigma$ in Y. Then, $Y = \left[\sum_{k=1}^{n} \left(y_{rk}^{\frac{1+\sigma}{\sigma}} \Pi_{k\neq r} y_{hk}^{\alpha_h} \right)^{\sigma} \right]^{\frac{1}{\sigma}} = \left[\sum_{k=1}^{n} y_{rk}^{1+\sigma} \left(\Pi_{k\neq r} y_{hk}^{\alpha_h}\right)^{\sigma}\right]^{\frac{1}{\sigma}}$. By taking the log, we have $\log Y = (1/\sigma) \log \sum_{k=1}^{n} y_{rk}^{1+\sigma} \left(\Pi_{k\neq r} y_{hk}^{\alpha_h}\right)^{\sigma}$. Then,

$$\lim_{\sigma\to 0} \log Y = \lim_{\sigma\to 0} \frac{1}{\sigma} \log \sum_{k=1}^{n} y_{rk}^{1+\sigma} \left(\Pi_{k\neq r} y_{hk}^{\alpha_h}\right)^{\sigma} = \lim_{\sigma\to 0} \frac{\sum_{k=1}^{n} \left[\frac{d y_{rk}^{1+\sigma}}{d\sigma} (\Pi_{k\neq r} y_{hk}^{\alpha_h})^{\sigma} + y_{rk}^{1+\sigma} \frac{d(\Pi_{h\neq r} y_{hk}^{\alpha_h})^{\sigma}}{d\sigma} \right]}{\sum_{k=1}^{n} y_{rk}^{1+\sigma} \left(\Pi_{k\neq r} y_{hk}^{\alpha_h}\right)^{\sigma}}.$$

Let $Z = y_{rk}^{1+\sigma}$ and $T = (\Pi_{h\neq r} y_{hk}^{\alpha_h})^{\sigma}$. Then, $\log Z = (1+\sigma)\log y_{rk} \Rightarrow Z'/Z = \log y_{rk} \iff Z' = y_{rk}^{1+\sigma} \log y_{rk}$ and $\log T = \sigma \log \Pi_{h\neq r} y_{hk}^{\alpha_h} \Rightarrow T'/T = \log \Pi_{h\neq r} y_{hk}^{\alpha_h} \iff T' = (\Pi_{h\neq r} y_{hk}^{\alpha_h})^{\sigma} \log \Pi_{h\neq r} y_{rk}^{\alpha_h}$. Substitute these into the above limit expression:

$$\begin{aligned}
& \lim_{\sigma\to 0} \frac{\sum_{k=1}^{n} \left[y_{rk}^{1+\sigma} (\log y_{rk}) (\Pi_{k\neq r} y_{hk}^{\alpha_h})^{\sigma} + y_{rk}^{1+\sigma} (\Pi_{h\neq r} y_{hk}^{\alpha_h})^{\sigma} \log \Pi_{h\neq r} y_{rk}^{\alpha_h} \right]}{\sum_{k=1}^{n} y_{rk}^{1+\sigma} \left(\Pi_{k\neq r} y_{hk}^{\alpha_h}\right)^{\sigma}} \\
= & \frac{\sum_{k=1}^{n} \left[y_{rk} \log y_{rk} + y_{rk} \log \Pi_{h\neq r} y_{hk}^{\alpha_h} \right]}{\sum_{k=1}^{n} y_{rk}} \\
= & \sum_{k=1}^{n} \left[\log y_{rk}^{y_{rk}} (\Pi_{h\neq r} y_{hk}^{\alpha_h})^{y_{rk}} \right] = \log \Pi_{k=1}^{n} y_{rk}^{y_{rk}} (\Pi_{h\neq r} y_{hk}^{\alpha_h})^{y_{rk}}.
\end{aligned}$$

Therefore, $Y \to \Pi_{k=1}^{n} y_{rk}^{y_{rk}} (\Pi_{h\neq r} y_{hk}^{\alpha_h})^{y_{rk}}$ as $\sigma \to 0$. ■

Proof of Lemma 2. Now, $r = t+1$, each y_{rk} for any k is not a controllable variable. We maximize (6) with $y_{1k}, y_{2k}, \ldots, y_{tk}$ for all $k = 1, \ldots, n$ such that $\sum_{k=1}^{n} y_{hk} = 1$ for all $h = 1, \ldots, t$, $1 \le t \le q-1$. The Lagrange function is defined as

$$\tilde{L} = \Pi_{k=1}^{n} \left(y_{rk} \Pi_{h\neq r} y_{hk}^{\alpha_h} \right)^{y_{rk}} + \sum_{h=1}^{t} \mu_h \left[1 - \sum_{k=1}^{n} y_{hk} \right].$$

Note that $\Pi_{k=1}^{n}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}} = \Pi_{k\neq j,l}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}}(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}(y_{rl}\Pi_{h\neq r}y_{hl}^{\alpha_h})^{y_{rl}}$. The first-order conditions are

$$\frac{\partial\tilde{L}}{\partial y_{sj}} = \Pi_{k\neq j,l}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}}\, y_{rj}(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}-1}y_{rj}\Pi_{h\neq r,s}y_{hj}^{\alpha_h}\alpha_s y_{sj}^{\alpha_s-1}(y_{rl}\Pi_{h\neq r}y_{hl}^{\alpha_h})^{y_{rl}} - \mu_s = 0$$

$$\frac{\partial\tilde{L}}{\partial y_{sl}} = \Pi_{k\neq j,l}(y_{rk}\Pi_{h\neq r}y_{hk}^{\alpha_h})^{y_{rk}}\,(y_{rj}\Pi_{h\neq r}y_{hj}^{\alpha_h})^{y_{rj}}y_{rl}(y_{rl}\Pi_{h\neq r}y_{hl}^{\alpha_h})^{y_{rl}-1}y_{rl}\Pi_{h\neq r,s}y_{hl}^{\alpha_h}\alpha_s y_{sl}^{\alpha_s-1} - \mu_s = 0$$

$$\frac{\partial\tilde{L}}{\partial\mu_s} = 1 - \sum_{k=1}^{n} y_{sk} = 0$$

for $s = 1,\dots,t$ and $j,l = 1,\dots,n$. From any $\partial\tilde{L}/\partial y_{sj}$ and $\partial\tilde{L}/\partial y_{sl}$, we have

$$\frac{y_{sj}}{y_{sl}} = \frac{y_{rj}}{y_{rl}} \iff y_{sl} = \frac{y_{rl}}{y_{rj}}y_{sj}.$$

By substituting y_{sl} for all $l = 1,\dots,n$ into $\partial\tilde{L}/\partial\mu_s$ and solving it for y_{sj}, we obtain $y_{sj} = y_{rj}/\sum_{k=1}^{n} y_{rk} = y_{rj}$ for $s = 1,\dots,t$ and $j = 1,\dots,n$ because $\sum_{k=1}^{n} y_{rk} = 1$. By substituting these expressions into (6), we have $\Pi_{k=1}^{n}(y_{rk}^{1+\sum_{g=1}^{t}\alpha_g}\Pi_{h=t+1,h\neq r}^{q}y_{hk}^{\alpha_h})^{y_{rk}}$. These correspond respectively to (4) and (5) when $\alpha_r = (1+\rho)/\rho$ and $\sum_{k=1}^{n} y_{rk} = 1$ for $r = t+1$, as $\rho \to 0$, by a similar calculation to that used to derive (6). ■

Proof of Lemma 3. Suppose that y_{rk} is added to the controllable variables. Since (7) is a Cobb-Douglass function form, $y_{rj} = 1$ with the highest $\Pi_{h=t+2}^{q}y_{hk}^{\alpha_h}$ and the other $y_{rl} = 0$ are required to maximize (7) from (8). We check that this solution in Lemma 2 does not include any indeterminate form in the limit. In fact, we have $\lim_{y_{rk}\to 0} y_{rk}^{y_{rk}(1+\sum_{g=1}^{t}\alpha_g)} = 1$

in (7) because

$$\begin{aligned}
\lim_{y_{rk}\to 0} \log y_{rk}^{y_{rk}(1+\sum_{g=1}^{t}\alpha_g)} &= \lim_{y_{rk}\to 0} \frac{\log y_{rk}^{(1+\sum_{g=1}^{t}\alpha_g)}}{1/y_{rk}} \\
&= \lim_{y_{rk}\to 0} \frac{(1+\sum_{g=1}^{t}\alpha_g) y_{rk}^{\sum_{g=1}^{t}\alpha_g} / y_{rk}^{1+\sum_{g=1}^{t}\alpha_g}}{-1/y_{rk}^2} \\
&= \lim_{y_{rk}\to 0} \frac{(1+\sum_{g=1}^{t}\alpha_g)/y_{rk}}{-1/y_{rk}^2} = 0
\end{aligned}$$

by using $\lim_{x\to+0} \log x^x = \lim_{x\to+0} x\log x = \lim_{x\to+0} \frac{\log x}{1/x} = \lim_{x\to+0} \frac{(\log x)'}{(1/x)'} = \lim_{x\to+0} \frac{1/x}{-1/x^2} = 0$. Therefore, Lemma 3 is obtained. ■

Proof of Lemma 4. When $\rho < 0$, noting that $1 - \rho\sum_{g=1}^{t}\alpha_g > 0$ and $\rho\alpha_{t+1} > 0$,

$$Y^* = \frac{1}{\left[\sum_{k=1}^{n} \left(\Pi_{h=t+1}^{q} y_{hk}^{\alpha_h}\right)^{\frac{\rho}{1-\rho\sum_{g=1}^{t}\alpha_g}}\right]^{\frac{1-\rho\sum_{g=1}^{t}\alpha_g}{-\rho}}} = \frac{1}{\left[\sum_{k=1}^{n} y_{t+1k}^{\frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g}} \left(\Pi_{h=t+2}^{q} y_{hk}^{\alpha_h}\right)^{\frac{\rho}{1-\rho\sum_{g=1}^{t}\alpha_g}}\right]^{\frac{1-\rho\sum_{g=1}^{t}\alpha_g}{-\rho}}}$$

From this form, it is sufficient to minimize the expression inside the brackets in the denominator. If $\frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g} > 1$, that is, $\rho\sum_{g=1}^{t+1}\alpha_g > 1$, in $y_{t+1k}^{\frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g}}$ in the brackets, the expression inside the brackets in Y^* is minimized by the interior solution because the expression inside the brackets is convex. The Lagrange function is defined as

$$\hat{L} = \sum_{k=1}^{n} y_{t+1k}^{\frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g}} \left(\Pi_{h=t+2}^{q} y_{hk}^{\alpha_h}\right)^{\frac{\rho}{1-\rho\sum_{g=1}^{t}\alpha_g}} + \nu\left[1 - \sum_{k=1}^{n} y_{t+1k}\right].$$

The first-order conditions are

$$\frac{\partial\hat{L}}{\partial y_{t+1j}} = \frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g} y_{t+1j}^{\frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g}-1} \left(\Pi_{h=t+2}^{q} y_{hj}^{\alpha_h}\right)^{\frac{\rho}{1-\rho\sum_{g=1}^{t}\alpha_g}} - \nu = 0$$

$$\frac{\partial L}{\partial\nu} = 1 - \sum_{k=1}^{n} y_{t+1k} = 0$$

for $j = 1, \ldots, n$. From any $\partial\hat{L}/\partial y_{t+1j}$ and $\partial\hat{L}/\partial y_{t+1l}$, we have

$$\frac{y_{t+1j}}{y_{t+1l}} = \left(\frac{\Pi_{h=t+2} y_{hj}^{\alpha_h}}{\Pi_{h=t+2} y_{hl}^{\alpha_h}}\right)^{\frac{\rho}{1-\rho\sum_{g=1}^{t+1}\alpha_g}}.$$

By substituting y_{t+1l} for all $l = 1, \ldots, n$ into $\partial L/\partial\nu$ and solving it for y_{t+1j}, we have (9) in the lemma, like the calculation in the proof of Lemma 1. Moreover, by substituting the solution into Y^*, we have (10) in the case of 1 in the lemma.

If $\frac{\rho\alpha_{t+1}}{1-\rho\sum_{g=1}^{t}\alpha_g} \leq 1$, that is, $\rho\sum_{g=1}^{t+1}\alpha_g \leq 1$, the expression inside the brackets in Y^* is minimized by the corner solution because the expression inside the brackets is concave. Note that because $\rho/(1-\rho\sum_{g=1}^{t}\alpha_g) < 0$, each $(\Pi_{h=t+2}^{q} y_{hk}^{\alpha_h})^{\rho/(1-\rho\sum_{g=1}^{t}\alpha_g)}$ is the denominator in the brackets in the denominator of Y^*. Then, $y_{t+1j} = 1$ with the highest $\Pi_{h=t+2}^{q} y_{hk}^{\alpha_h}$ and the other $y_{t+1l} = 0$ maximize Y^*. Then, $Y^{**} = \Pi_{h=t+2}^{q} y_{hk}^{\alpha_h}$. ■

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